

THE BI-NORMAL FIELDS ON SPACELIKE SURFACES IN \mathbb{R}_1^4

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Abstract

A normal field on a spacelike surface in R_1^4 is called bi-normal if K^ν , the determinant of Weingarten map associated with ν , is zero. In this paper we give a relationship between the spacelike pseudo-planar surfaces and spacelike pseudo-umbilical surfaces, then study the bi-normal fields on spacelike ruled surfaces and spacelike surfaces of revolution.

0 Introduction

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a bi-regular parametric curve. Along this curve, the vector field defined by

$$\mathbf{b} = \frac{1}{|\alpha' \times \alpha''|}(\alpha' \times \alpha'')$$

is called the bi-normal field of α . A bi-normal vector can be seen as a direction whose the corresponding height function has a degenerate (non-Morse) critical point.

Let M be a regular surface in \mathbb{R}_1^4 (or \mathbb{R}^4) and $f_{\mathbf{v}}$ be the height function on M associated with a direction \mathbf{v} . By analogy with the case of curves in \mathbb{R}^3 , a direction \mathbf{v} is called a bi-normal direction of M at a point p if the height function $f_{\mathbf{v}}$ has a degenerate singularity at p . The height function $f_{\mathbf{v}}$ having a degenerate singularity means that its hessian is singular.

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Given a normal field ν on M and denoted by S^ν the shape operator associated with ν . The ν -Gauss curvature K^ν of M is defined by $K^\nu = \det S^\nu$. The eigenvalues k_1^ν and k_2^ν of the shape operator S^ν are called ν -principal curvatures. The ν -mean curvature of M is defined by

$$H^\nu = \frac{1}{2} \text{trace} S^\nu = \frac{1}{2} (k_1^\nu + k_2^\nu).$$

A point $p \in M$ is called ν -umbilic if $k_1^\nu(p) = k_2^\nu(p) = k$ and is called ν -flat if $k_1^\nu(p) = k_2^\nu(p) = 0$. If there exists a normal field ν on M such that M is ν -umbilic (ν -flat) then M is called pseudo-umbilical (pseudo-flat) surface. M is called umbilic if it is ν -umbilic for all normal fields ν . M is called maximal if $H^\nu = 0$ for all normal fields ν .

It is easy to show that $\nu(p)$ is bi-normal direction of M at p if $\det S^{\nu(p)} = 0$ i.e. either $k_1^\nu(p) = 0$ or $k_2^\nu(p) = 0$. Such a point is called ν -planar and a direction belonged to the kernel of $S^{\nu(p)}$ is said to be asymptotic. The normal field ν of M is called bi-normal field if for each $p \in M$, $\nu(p)$ is bi-normal direction of M at p . If there exists a bi-normal field on M then M is called pseudo-planar surface, in the case each normal field is bi-normal M is called planar surface. For everything concerning to these notions in more detail, we refer the reader to [9], [8], [16], [17], [19] and references therein.

The existence of bi-normal direction on surfaces in \mathbb{R}^4 has been studied by several authors (e.g. [4], [16], [17], [19], [21], [14], [22], ...). Little ([14], Theorem 1.3(b), 1969) showed that a surface whose all normal fields are bi-normal if and only if it is a ruled developable surface. In 1995, D.K.H. Mochida et. al ([16], Corollary 4.3 repeated at [21], (2010)) showed that a surface admitting two bi-normal fields if and only if it is strictly locally convex. These results was expanded to surfaces of codimension two in \mathbb{R}^{n+2} by them [17] in 1999. These methods are used later by M.C. Romero-Fuster and F. Sánchez-Brigas ([19], Theorem 3.4, 2002) to study the umbilicity of surfaces.

The first section of this paper shows that there exist pseudo-planar surfaces are not pseudo-umbilic, defines the number of bi-normal fields on the pseudo-umbilical surfaces and gives some interesting corollaries.

In the second of this paper we show that each point on the spacelike ruled surfaces admits either one or all bi-normal directions, a spacelike ruled surface is pseudo-umbilic iff umbilic.

In the third section of this paper we show that the spacelike surfaces of revolution admit exactly two bi-normal fields whose asymptotic fields respectively are orthogonal. Therefore, they are pseudo-umbilic but not umbilic.

The final section of this paper shows that the number of bi-normal fields on the rotational spacelike surface whose meridians lie in two-dimension space are depended on the properties of its meridian.

1 Bi-normal Fields on Pseudo-umbilical Surfaces

For the surfaces in \mathbb{R}^4 Romero Fuster [19] showed that pseudo-umbilical surfaces are pseudo-planar; moreover, their two asymptotic fields are orthogonal. These results are also true for spacelike surfaces in \mathbb{R}_1^4 , and I would like to show it here. Notice that there exist the pseudo-planar spacelike surfaces are not pseudo-umbilic, let see Example 1.2 and Example 1.3. We have the similar example for surfaces in \mathbb{R}^4 .

The following theorem shows that the pseudo-umbilical spacelike surfaces are pseudo-planar and gives us the number of bi-normal fields on them.

THEOREM 1.1. *Let M be a spacelike surface in \mathbb{R}_1^4 . If M is pseudo-umbilic (not pseudo-flat) then it admits either one or two bi-normal fields. Moreover, M admits only one bi-normal field iff it is umbilic.*

PROOF. Suppose that M is ν -umbilic (not ν -flat). Let \mathbf{n} be a normal field on M such that $\{\mathbf{X}_u, \mathbf{X}_v, \nu, \mathbf{n}\}$ is linearly independent and k is ν -principal curvature. Given a normal field \mathbf{B} , then we have the following interpretation

$$\mathbf{B} = \lambda\nu + \mu\mathbf{n},$$

where λ, μ are smooth functions on M . Suppose that the coefficients of the first fundamental form of M satisfy

$$g_{11} = g_{22} = \varphi, g_{12} = 0,$$

then we have

$$S^{\mathbf{B}} = \lambda S^{\nu} + \mu S^{\mathbf{n}} = \lambda k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\mu}{\varphi} \begin{bmatrix} b_{11}^{\mathbf{n}} & b_{12}^{\mathbf{n}} \\ b_{12}^{\mathbf{n}} & b_{22}^{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} \frac{\mu}{\varphi} b_{11}^{\mathbf{n}} + \lambda k & \frac{\mu}{\varphi} b_{12}^{\mathbf{n}} \\ \frac{\mu}{\varphi} b_{12}^{\mathbf{n}} & \frac{\mu}{\varphi} b_{22}^{\mathbf{n}} + \lambda k \end{bmatrix}.$$

Therefore,

$$\begin{aligned} K^{\mathbf{B}} &= \gamma^2 (b_{11}^{\mathbf{n}} b_{22}^{\mathbf{n}} - (b_{12}^{\mathbf{n}})^2) + \lambda \gamma k (b_{11}^{\mathbf{n}} + b_{22}^{\mathbf{n}}) + \lambda^2 k^2, \\ K^{\mathbf{B}} = 0 &\Leftrightarrow \gamma^2 (b_{11}^{\mathbf{n}} b_{22}^{\mathbf{n}} - (b_{12}^{\mathbf{n}})^2) + \lambda \gamma k (b_{11}^{\mathbf{n}} + b_{22}^{\mathbf{n}}) + \lambda^2 k^2 = 0, \end{aligned} \quad (1)$$

where $\gamma = \frac{\mu}{\varphi}$. Since ν is not bi-normal, $\mu \neq 0$. Then the equation (1) can be rewrote by

$$\left(\frac{\lambda k}{\gamma} \right)^2 + (b_{11}^{\mathbf{n}} + b_{22}^{\mathbf{n}}) \frac{\lambda k}{\gamma} + b_{11}^{\mathbf{n}} b_{22}^{\mathbf{n}} - (b_{12}^{\mathbf{n}})^2 = 0. \quad (2)$$

It is from

$$(b_{11}^{\mathbf{n}} - b_{22}^{\mathbf{n}})^2 + 4(b_{12}^{\mathbf{n}})^2 \geq 0 \quad (3)$$

that the equation (2) has at least one or at most two solutions. That means M admits at least one or at most two bi-normal fields.

M admits only one bi-normal field if and only if $b_{11}^{\mathbf{n}} = b_{22}^{\mathbf{n}}$ and $b_{12}^{\mathbf{n}} = 0$. Which means that M is \mathbf{n} -umbilic. Then the Lemma 4.1 in [2] shows that M is umbilic. \square

The following example gives a spacelike surface admitting one bi-normal field but not pseudo-umbilic.

EXAMPLE 1.2. Let M be a surface given by following parameterization

$$\mathbf{X}(u, v) = (\cos u(1 + v), \sin u(1 + v), \sinh u, \cosh u), \quad u, v \in \mathbb{R}. \quad (4)$$

The coefficients of the first fundamental form of M are determined by

$$g_{11} = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = v^2 + 2 > 0, \quad g_{12} = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0, \quad g_{22} = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = 1.$$

Therefore, M is a spacelike surface. Let $\mathbf{n} = (n^1, n^2, n^3, n^4)$ be a normal field on M . That means

$$\langle \mathbf{X}_u, \mathbf{n} \rangle = 0 \Leftrightarrow n^1 \cos u + n^2 \sin u = 0, \quad (5)$$

$$\langle \mathbf{X}_v, \mathbf{n} \rangle = 0 \Leftrightarrow -n^1 \sin u(1 + v) + n^2 \cos u(1 + v) + n^3 \cosh u - n^4 \sinh u = 0. \quad (6)$$

Using (5) the coefficients of the second fundamental form associated with \mathbf{n} are

$$b_{11}^{\mathbf{n}} = n^3 \sinh u - n^4 \cosh u, \quad b_{12}^{\mathbf{n}} = -n^1 \sin u + n^2 \cos u, \quad b_{22}^{\mathbf{n}} = 0.$$

We have

$$\det(b_{ij}^{\mathbf{n}}) = (b_{12}^{\mathbf{n}})^2.$$

So,

$$K^{\mathbf{n}} = 0 \Leftrightarrow -n^1 \sin u + n^2 \cos u = 0. \quad (7)$$

Connecting (5), (6) and (7) we imply that \mathbf{n} is a bi-normal field on M if and only if

$$n^1 = n^2 = 0, \quad n^3 \cosh u - n^4 \sinh u = 0.$$

That means M admits only one bi-normal field

$$\mathbf{B} = (0, 0, \sinh u, \cosh u).$$

It is a unit timelike normal field. Since

$$S^{\mathbf{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

M is not \mathbf{B} -flat.

On the other hand the \mathbf{n} -principal curvatures are the solutions of the following equation

$$\det((b_{ij}^{\mathbf{n}}) - \lambda(g_{ij})) = 0 \Leftrightarrow (v^2 + 2)\lambda^2 - \lambda b_{11}^{\mathbf{n}} - (b_{12}^{\mathbf{n}})^2 = 0.$$

Therefore, M is \mathbf{n} -umbilic if and only if $b_{11}^{\mathbf{n}} = b_{12}^{\mathbf{n}} = 0$. Which doesn't take place, by connecting (5), (6) and (7). So, M is not pseudo-umbilic.

Even when M admits two bi-normal fields, it is not pseudo-umbilic. Let see the following example.

EXAMPLE 1.3. Let M be a surface given by following parameterization

$$\mathbf{X}(u, v) = (e^{2u} \cos v, e^{2u} \sin v, e^{-u} \cosh v, e^{-u} \sinh v), \quad u > 1, \quad v \in (0, 2\pi).$$

It is easy to show that M is spacelike and $\{\mathbf{n}_1, \mathbf{n}_2\}$ is a frame of the variable normal bundle, where

$$\begin{aligned} \mathbf{n}_1 &= -\frac{1}{\sqrt{g_{11}}} (e^{-u} \cos v, e^{-u} \sin v, 2e^{2u} \cosh v, 2e^{2u} \sinh v), \\ \mathbf{n}_2 &= \frac{1}{\sqrt{g_{22}}} (-e^{-u} \sin v, e^{-u} \cos v, e^{2u} \sinh v, e^{2u} \cosh v). \end{aligned}$$

The coefficients of the second fundamental form associated with \mathbf{n}_1 and \mathbf{n}_2 are

$$\begin{aligned} b_{11}^{\mathbf{n}_1} &= -\frac{6e^u}{\sqrt{g_{11}}}, \quad b_{12}^{\mathbf{n}_1} = 0, \quad b_{22}^{\mathbf{n}_1} = -\frac{e^u}{\sqrt{g_{11}}}; \\ b_{11}^{\mathbf{n}_2} &= 0, \quad b_{12}^{\mathbf{n}_2} = \frac{3e^u}{\sqrt{g_{22}}}, \quad b_{22}^{\mathbf{n}_2} = 0. \end{aligned}$$

Therefore, both \mathbf{n}_1 and \mathbf{n}_2 are not bi-normal. Fore each normal field \mathbf{n} on M we have the following interpretation

$$\mathbf{n} = \mathbf{n}_1 + \mu \mathbf{n}_2 \tag{8}$$

and

$$(b_{ij}^{\mathbf{n}}) = \begin{bmatrix} b_{11}^{\mathbf{n}_1} & \mu b_{12}^{\mathbf{n}_2} \\ \mu b_{12}^{\mathbf{n}_2} & b_{22}^{\mathbf{n}_1} \end{bmatrix}.$$

So

$$K^{\mathbf{n}} = \frac{b_{11}^{\mathbf{n}_1} b_{22}^{\mathbf{n}_1} - \mu^2 (b_{12}^{\mathbf{n}_2})^2}{g_{11} g_{22}}.$$

Since $b_{11}^{\mathbf{n}_1} b_{22}^{\mathbf{n}_1} > 0$ and $b_{12}^{\mathbf{n}_2} \neq 0$, M admits exactly two bi-normal fields.

On the other hand the \mathbf{n} -principal curvatures of M are solutions of the following equation

$$\det((b_{ij}^{\mathbf{n}}) - \lambda(g_{ij})) = 0 \Leftrightarrow g_{11} g_{22} \lambda^2 + \left(e^u \sqrt{g_{11}} + \frac{6e^u g_{22}}{\sqrt{g_{11}}} \right) \lambda + \frac{6e^{2u}}{g_{11}} - \frac{9e^{2u} \mu^2}{g_{22}} = 0, \tag{9}$$

where λ is the variable. Since

$$\frac{1}{g_{11}} \left[(2e^{4u} - 7e^{-2u})^2 \right] + 36g_{11} \mu^2 > 0, \quad \forall \mu,$$

for each normal field \mathbf{n} , M is not \mathbf{n} -umbilic. It means that M is not pseudo-umbilic.

COROLLARY 1.4. *If M is umbilic then M is pseudo-flat.*

COROLLARY 1.5. *Let M be a surface contained in the a pseudo-sphere (Hyperbolic or de Sitter). Then the following statements are equivalent.*

- (1) M is umbilic,
- (2) M admits only one bi-normal field,
- (3) M is contained in a hyperplane.

COROLLARY 1.6. *The following statements are equivalent.*

- (1) M is locally umbilic.
- (2) M is locally contained in the intersection of a Hyperbolic (or de Sitter) with a hyperplane.
- (3) M locally admits only one bi-normal field \mathbf{B} and ν -umbilic (not ν -flat), for some normal field ν .

COROLLARY 1.7. *If M admits only one bi-normal field \mathbf{B} (not \mathbf{B} -flat) then it is not pseudo-umbilic.*

REMARK 1.8. The results in [19] are also true for the spacelike surfaces in \mathbb{R}_1^4 . So that the following statements are equivalent:

- (1) M has two everywhere defined orthogonal asymptotic fields,
- (2) M is pseudo-umbilic,
- (3) The normal curvature of M vanishes at every point,
- (4) All points of M are semi-umbilic.

2 Bi-normal Fields on Ruled Spacelike Surface in \mathbb{R}_1^4

The notion of ruled surface in \mathbb{R}^4 have been introduced by Lane in [13]. It is similar to ruled (spacelike) surface in \mathbb{R}_1^4 and can be introduced by the similar way. A surface M in \mathbb{R}_1^4 is called ruled if through every point of M there is a straight line that lies on M . We have a local parameterization of M

$$\mathbf{X}(u, t) = \alpha(t) + uW(t), \quad t \in I \subset \mathbb{R}, u \in \mathbb{R}, \quad (10)$$

where $\alpha(t)$ is a differential curve in \mathbb{R}_1^4 and $W(t)$ is a smooth vector field along $\alpha(t)$.

A ruled surface M is called developable if its Gaussian curvature identifies zero.

It is from $\mathbf{X}_u = W(t)$, $\mathbf{X}_t(0, t) = \alpha'(t)$ and M is spacelike that both $W(t)$ and $\alpha'(t)$ are spacelike. We can assume that $|W| = |\alpha'| = 1$ and $\langle W, \alpha' \rangle = 0$.

The coefficients of the first fundamental form of M are

$$g_{11} = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = \langle \alpha', \alpha' \rangle + 2t\langle \alpha', W' \rangle + t^2\langle W', W' \rangle,$$

$$g_{12} = \langle \mathbf{X}_u, \mathbf{X}_t \rangle = 0, \quad g_{22} = \langle \mathbf{X}_t, \mathbf{X}_t \rangle = 1.$$

Since M is spacelike, $\langle W', W' \rangle > 0$.

Let \mathbf{n} be a normal field on M , the coefficients of the second fundamental form associated \mathbf{n} are defined as following

$$b_{11}^{\mathbf{n}} = \langle \mathbf{X}_{uu}, \mathbf{n} \rangle = \langle \alpha'', \mathbf{n} \rangle + t\langle W'', \mathbf{n} \rangle, \quad b_{12}^{\mathbf{n}} = \langle \mathbf{X}_{ut}, \mathbf{n} \rangle = \langle W', \mathbf{n} \rangle,$$

$$b_{22}^{\mathbf{n}} = \langle \mathbf{X}_{tt}, \mathbf{n} \rangle = 0.$$

Therefore,

$$S^{\mathbf{n}} = (g_{ij})^{-1} \cdot (b_{ij}^{\mathbf{n}}) = \frac{1}{g_{11}} \begin{bmatrix} b_{11}^{\mathbf{n}} & b_{12}^{\mathbf{n}} \\ b_{12}^{\mathbf{n}}g_{11} & 0 \end{bmatrix}, \quad K^{\mathbf{n}} = -\frac{(b_{12}^{\mathbf{n}})^2}{g_{11}}. \quad (11)$$

So,

$$K^{\mathbf{n}} = 0 \Leftrightarrow b_{12}^{\mathbf{n}} = 0 \Leftrightarrow \langle W', \mathbf{n} \rangle = 0. \quad (12)$$

The following proposition gives us the number of bi-normal directions at each point on a ruled surface.

PROPOSITION 2.1. *Let M be a ruled spacelike surface given by (10), we then have:*

1. *at the point such that $\{\alpha', W, W'\}$ is linearly dependent each normal vector is bi-normal direction;*
2. *at the point such that $\{\alpha', W, W'\}$ is linearly independent M admits only one bi-normal direction.*
3. *M is pseudo-umbilic if and only if umbilic.*

PROOF.

1. Since

$$\langle \alpha', W \rangle = 0, \quad \langle W', W \rangle = 0$$

and $\{\alpha', W, W'\}$ is linearly dependent, $W' \in T_p M$. Therefore, by using (12), we imply that each normal vector on M is bi-normal direction.

2. Since

$$K\mathbf{n} = 0 \Leftrightarrow \begin{cases} \langle \mathbf{n}, \mathbf{X}_u \rangle = 0, \\ \langle \mathbf{n}, \mathbf{X}_v \rangle = 0, \\ \langle \mathbf{n}, W' \rangle = 0, \end{cases} \Leftrightarrow \begin{cases} \langle \mathbf{n}, \alpha' \rangle = 0, \\ \langle \mathbf{n}, W \rangle = 0, \\ \langle \mathbf{n}, W' \rangle = 0, \end{cases}$$

\mathbf{n} is an unit bi-normal direction if and only if

$$\mathbf{n} = \frac{\alpha' \wedge W \wedge W'}{|\alpha' \wedge W \wedge W'|}.$$

It is followed from the fact that α', W, W' are spacelike that the unique unit bi-normal direction on M is timelike.

3. Since M admits only one bi-normal field, it is followed Theorem 1.1 that M is pseudo-umbilic iff umbilic.

□

REMARK 2.2. 1. The Proposition 2.1 is also true for the ruled surfaces in \mathbb{R}^4 .

2. Using the Gauss equation we can show that the Gaussian curvature of a spacelike surface in \mathbb{R}_1^4 can be defined by sum of K^{e_1} and K^{e_2} , where $\{e_1, e_2\}$ is an orthogonal frame of normal bundle of surface. Therefore, a ruled spacelike surface is developable iff $\{\alpha', W, W'\}$ is linearly dependent.

3. Similarly the results on the surfaces in \mathbb{R}^4 (see [14]), it is easy to show that if a spacelike surface M is planar and the causal character of its ellipse curvature (see [9]) is invariant then M is a ruled developable surface.

Lane [13] showed that if a ruled surface in \mathbb{R}^4 is minimal then it is contained in a hyperplane and of course it is either plane or helicoid. We have the same results for the maximal ruled spacelike surfaces in \mathbb{R}_1^4 . That means a ruled spacelike surface in \mathbb{R}_1^4 is maximal if and only if it is maximal in a spacelike hyperplane.

3 Bi-normal Fields on Spacelike Surfaces of Revolution

Let C be a spacelike curve in $\text{span}\{e_1, e_2, e_4\}$ parametrized by arc-length,

$$z(u) = (f(u), g(u), 0, \rho(u)), \quad \rho(u) > 0, \quad u \in I.$$

The orbit of C under the action of the orthogonal transformations of \mathbb{R}_1^4 leaving the spacelike plane Oxy ,

$$A_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{bmatrix}, \quad v \in \mathbb{R},$$

is a surface given by

$$[\text{RH}] \quad \mathbf{X}(u, v) = (f(u), g(u), \rho(u) \sinh v, \rho(u) \cosh v), \quad u \in I, \quad v \in \mathbb{R}. \quad (13)$$

The coefficients of the first fundamental form of $[\text{RH}]$ are

$$g_{11} = (f'(u))^2 + (g'(u))^2 - (\rho'(u))^2 = 1, \quad g_{12} = 0, \quad g_{22} = (\rho(u))^2 > 0.$$

It follows that $[\text{RH}]$ is a spacelike surface, which is called the *spacelike surface of revolution of hyperbolic type* in \mathbb{R}_1^4 . From now on we always assume that $f' \neq 0, g' \neq 0$ and $\rho' \neq 0$.

PROPOSITION 3.1. *Suppose that $f'g'' - f''g' \neq 0$, we then have:*

- (a) *$[\text{RH}]$ admits exactly two bi-normal fields and its asymptotic fields are orthogonal,*
- (b) *There exists only one normal field ν satisfying that $[\text{RH}]$ is ν -umbilic.*

PROOF. (a) Let $\mathbf{n} = (n_1, n_2, n_3, n_4)$ be a normal field on M , we have

$$\langle \mathbf{X}_u, \mathbf{n} \rangle = 0, \quad \langle \mathbf{X}_v, \mathbf{n} \rangle = 0.$$

That means

$$n_1 f' + n_2 g' + n_3 \rho' \sinh v - n_4 \rho' \cosh v = 0, \quad \rho(n_3 \cosh v - n_4 \sinh v) = 0. \quad (14)$$

Since (14),

$$b_{12}^{\mathbf{n}} = \langle \mathbf{X}_{uv}, \mathbf{n} \rangle = \rho' (n_3 \cosh v - n_4 \sinh v) = 0.$$

Therefore,

$$\det(S^{\mathbf{n}}) = \frac{b_{11}^{\mathbf{n}} \cdot b_{22}^{\mathbf{n}}}{\rho^2}, \quad (15)$$

where $b_{ij}^{\mathbf{n}}$ are the coefficients of the second fundamental form associated with \mathbf{n} of $[\text{RH}]$.

On the other hand we have

$$\langle \mathbf{X}_u, \mathbf{X}_u \rangle = 1 \Rightarrow \langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle = 0,$$

$$\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 \Rightarrow \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle = -\langle \mathbf{X}_u, \mathbf{X}_{uv} \rangle = 0.$$

So, $\{\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}\}$ is linearly independent. Therefore, $b_{11}^{\mathbf{n}} = 0$ if and only if \mathbf{n} is parallel to

$$\mathbf{B}_1 = \mathbf{X}_u \wedge \mathbf{X}_v \wedge \mathbf{X}_{uu}.$$

It is easy to show that $b_{22}^{\mathbf{n}} = 0$ if and only if \mathbf{n} is parallel to $\mathbf{B}_2 = (-g', f', 0, 0)$. \mathbf{X}_v then is asymptotic field associated with \mathbf{B}_2 .

Since $f'g'' - f''g' \neq 0$, $\mathbf{B}_1, \mathbf{B}_2$ are linearly independent. So, [RH] admits exactly two bi-normal fields.

(b) Using base $\{\mathbf{X}_u, \mathbf{X}_v\}$ for tangent planes of [RH], we have

$$S^{\mathbf{B}_1} = \begin{bmatrix} 0 & 0 \\ 0 & -f'g'' + f''g' \end{bmatrix}, \quad S^{\mathbf{B}_2} = \begin{bmatrix} f'g'' - f''g' & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, [RH] is ν -umbilic, where $\nu = \mathbf{B}_1 - \mathbf{B}_2$. Remark 1.8 shows that the normal curvature of [RH] identifies zero, [RH] has two orthogonal asymptotic fields everywhere, and [RH] is semi-umbilic. \square

REMARK 3.2.

(a) If $f'g'' - f''g' = 0$ then M is contained in a hyperplane.

(b) It is similar to the spacelike surfaces of revolution of elliptic type.

4 Bi-normal Fields on The Rotational Spacelike Surfaces Whose Meridians Lie in Two-dimension planes

Romero Fuster et. al [17] showed that there always an open region of a generic, compact 2-manifold in \mathbb{R}^4 all whose points admit at least one bi-normal direction and at most n of them. This result is not true in the general case. This section gives a class of spacelike surfaces whose points can admit non, one, two or infine bi-normal directions. It is similar to them on \mathbb{R}^4 .

Let C be a spacelike curve contained in $\text{span}\{e_1, e_3\}$ and parametrized by

$$r(u) = (f(u), 0, g(u), 0), \quad u \in I,$$

and

$$A = \begin{bmatrix} \cos \alpha v & -\sin \alpha v & 0 & 0 \\ \sin \alpha v & \cos \alpha v & 0 & 0 \\ 0 & 0 & \cosh \beta v & \sinh \beta v \\ 0 & 0 & \sinh \beta v & \cosh \beta v \end{bmatrix}, \quad v \in [0, 2\pi),$$

such that

$$\alpha^2 f^2(u) - \beta^2 g^2(u) > 0,$$

be a orthogonal transformations of \mathbb{R}_1^4 , where $u \in J \subset \mathbb{R}$ and α, β are positive constants.

The orbit of C under the action of the orthogonal transformations A is a surface [RS] given by

$$\mathbf{X}(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cosh \beta v, g(u) \sinh \beta v). \quad (16)$$

The coefficients of the first fundamental form of [RS] are

$$g_{11} = (f')^2 + (g')^2 > 0, \quad g_{12} = 0, \quad g_{22} = \alpha^2 f^2 - \beta^2 g^2 > 0.$$

That means [RS] is spacelike. It is called *rotational spacelike surface whose meridians lie in two-dimension planes of type I*.

Choosing $\{\mathbf{n}_1, \mathbf{n}_2\}$ is an orthonormal frame field on [RS], where

$$\mathbf{n}_1 = \frac{1}{\sqrt{(f')^2 + (g')^2}} (g' \cos \alpha v, g' \sin \alpha v, -f' \cosh \beta v, -f' \sinh \beta v),$$

$$\mathbf{n}_2 = \frac{1}{\sqrt{\alpha^2 f^2 - \beta^2 g^2}} (-\beta g \sin \alpha v, \beta g \cos \alpha v, \alpha f \sinh \beta v, \alpha f \cosh \beta v),$$

then the coefficients of the second fundamental form associated to \mathbf{n}_1 and \mathbf{n}_2 are defined by

$$b_{11}^{\mathbf{n}_1} = \frac{f''g' - f'g''}{\sqrt{(f')^2 + (g')^2}}, \quad b_{12}^{\mathbf{n}_1} = 0, \quad b_{22}^{\mathbf{n}_1} = -\frac{\beta^2 f'g + \alpha^2 fg'}{\sqrt{(f')^2 + (g')^2}},$$

$$b_{11}^{\mathbf{n}_2} = 0, \quad b_{12}^{\mathbf{n}_2} = \frac{\alpha\beta(f'g - fg')}{\sqrt{\alpha^2 f^2 - \beta^2 g^2}}, \quad b_{22}^{\mathbf{n}_2} = 0,$$

respectively.

Let \mathbf{B} be a normal field on [RS], we have

$$\mathbf{B} = \lambda \mathbf{n}_1 + \mu \mathbf{n}_2,$$

where λ, μ are smooth functions on [RS]. Then we have

$$(b_{ij}^{\mathbf{B}}) = \lambda(b_{ij}^{\mathbf{n}_1}) + \mu(b_{ij}^{\mathbf{n}_2}) = \begin{bmatrix} \lambda b_{11}^{\mathbf{n}_1} & \mu b_{12}^{\mathbf{n}_2} \\ \mu b_{12}^{\mathbf{n}_2} & \lambda b_{22}^{\mathbf{n}_1} \end{bmatrix}.$$

So,

$$K^{\mathbf{B}} = \frac{\lambda^2 b_{11}^{\mathbf{n}_1} \cdot b_{22}^{\mathbf{n}_1} - \mu^2 (b_{12}^{\mathbf{n}_2})^2}{((f')^2 + (g')^2) (\alpha^2 f^2 - \beta^2 g^2)}.$$

Therefore,

- (a) \mathbf{n}_2 is bi-normal if and only if $f = cg$, where c is constant satisfying $\alpha^2 - c\beta^2 > 0$. Then $b_{22}^{\mathbf{n}_1} = 0$. So, \mathbf{n}_1 is also bi-normal. And it is easy to show that [RS] is a planar. That means [RS] is planar if and only if C is a line passing through the origin.
- (b) \mathbf{n}_1 is bi-normal if and only if either

$$f = cg + c_1 \text{ or } \alpha^2 fg' + \beta^2 f'g = 0, \quad (17)$$

where c, c_1 are constant. In this case, if $c_1 \neq 0$ then [RS] admits only one bi-normal field that is \mathbf{n}_1 . Therefore, [RS] admits only one bi-normal field if and only \mathbf{n}_1 is bi-normal and \mathbf{n}_2 is not bi-normal. Which takes place if and only if (17) is true and $c_1 \neq 0$. For example

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, \cosh v, \sinh v), \quad u > 1, \quad v \in (0, 2\pi).$$

- (c) [RS] does not admit any bi-normal field if and only if

$$-(f''g' - f'g'')(\beta^2 f'g + \alpha^2 fg') < 0 \text{ and } \alpha\beta(f'g - g'f) \neq 0.$$

For example

$$\mathbf{X}(u, v) = (u^2 \cos v, u^2 \sin v, u \cosh v, u \sinh v), \quad u > 1, \quad v \in (0, 2\pi).$$

- (d) [RS] admits exactly two bi-normal fields if and only if

$$-(f''g' - f'g'')(\beta^2 f'g + \alpha^2 fg') > 0 \text{ and } \alpha\beta(f'g - g'f) \neq 0.$$

For example

$$\mathbf{X}(u, v) = (e^{2u} \cos v, e^{2u} \sin v, e^{-u} \cosh v, e^{-u} \sinh v), \quad u > 1, \quad v \in (0, 2\pi).$$

It is similar to the rotational spacelike surfaces whose meridians lie in two-dimension planes of type II. This result is also true for the rotational surfaces whose meridians lie in two-dimension planes in \mathbb{R}^4 .

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